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Contoh 15

Sebuah massa benda kecil $m = 2$ kg dilekatkan pada ujung bawah suatu pegas elastis yang ujung atasnya tetap di mana konstanta pegasnya diketahui $k = 10$ N/m. Andaikan $y(t)$ perpindahan benda itu dari posisi kesetimbangannya, tentukan getaran bebas dari benda jika diberikan posisi awal $y(0) = 2$ m dengan kecepatan awal $y'(0) = -4$ m/dt dan anggap ada redaman terhadap kecepatan dengan konstanta redaman $c = 4$ N dt/m.

Diketahui : $m = 2$ kg, $y(0) = 2$ m, $c = 4$ N dt/m

$k = 10$ N/m, $y'(0) = -4$ m/dt

Ditanya : $y(t)$

Jawab : Persamaan ; Sesuai dengan persamaan pegas pada subbab 4.2.2 halaman 79, yakni :

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{1}{m}f(y) \quad \times m$$

$$m.y'' + c.y' + ky = 0$$

$$\Rightarrow 2y'' + 4y' + 10y = 0 \;\; :2, \quad y(0) = 2,\; y'(0) = -4$$

$$\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\{y\} = \mathcal{L}\{0\}$$

$$[s^2F(s) - sy(0) - y'(0)] + 2[sF(s) - y(0)] + 5F(s) = 0$$

$$s^2F(s) - 2s + 4 + 2sF(s) - 4 + 5F(s) = 0$$

$$F(s)[s^2 + 2s + 5] = 2s + 4 - 4$$

$$F(s) = \frac{2s}{s^2+2s+5} = \frac{2s}{(s+1)^2+2^2} = 2\frac{s}{(s+1)^2+2^2}$$

▶ Manipulasi agar menjadi bentuk fungsi dasar transformasi Laplace

$$F(s) = \frac{2(s+1)-2}{(s+1)^2+2^2} = 2\frac{(s+1)}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

$$= 2[e^{-t}.\cos 2t] - e^{-t}\sin 2t$$

$$= 2e^{-t}.\cos 2t - e^{-t}\sin 2t$$

$$y(t) = e^{-t}[2\cos 2t - \sin 2t]$$

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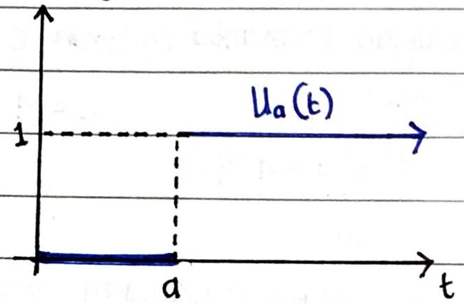
※ Contoh 16

Sketsa dan dapatkan transformasi Laplace dari $U_a(t)$

▶ Definisi $U_a(t)$.

$$U_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

Sehingga, sketsanya :



Berdasarkan teorema $\mathcal{L}\{f(t-a).U_a(t)\} = e^{-as}F(s)$

di mana $F(s) = \mathcal{L}\{f(t)\}$, maka

$\mathcal{L}\{U_a(t)\} = \mathcal{L}\{1\, U_a(t)\}$,

sehingga, $a = a$, $f(t-a) = f(t) = 1$

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{1\}$

$= \frac{1}{s}$

Maka, diperoleh : $\mathcal{L}\{U_a(t)\} = e^{-as} \cdot \frac{1}{s} = \frac{e^{-as}}{s}$ //

※ Contoh 17

Sketsa fungsi $f(t) = U_a(t) - U_b(t)$, $b > a$ dan tentukan Transformasi Laplacenya.

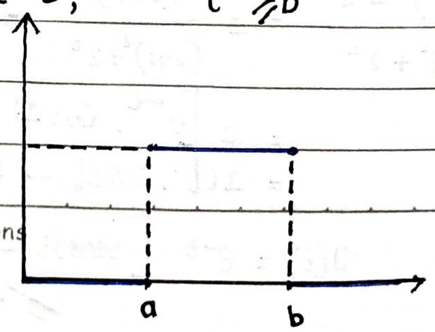
sesuai definisi :

$$U_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

$$U_b(t) = \begin{cases} 0, & t < b \\ 1, & t \geq b \end{cases}$$

$$f(t) = \begin{cases} 0, & t < a \\ 1, & a \leq t < b \\ 0, & t \geq b \end{cases}$$

Sehingga, sketsanya :



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$$\mathcal{L}\{f(t)\} = \mathcal{L}\{U_a(t) - U_b(t)\}$$

Sesuai contoh soal 16, maka

$$= \mathcal{L}\{U_a(t)\} - \mathcal{L}\{U_b(t)\}$$

$$= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$$

$$\mathcal{L}\{U_a(t) - U_b(t)\} = \frac{e^{-as} - e^{-bs}}{s} \quad //$$

※ Contoh 18

Dapatkan invers transformasi laplace dari $F(s) = \frac{e^{-3s}}{s^3}$.

※ Bentuk umum penyelesaian :

$$\mathcal{L}\{f(t-a).\,U_a(t)\} = e^{-as}F(s)$$

maka : $e^{-as}F(s) = e^{-3s} . \frac{1}{s^3}$

※ $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2} . \frac{2}{s^3}\right\}$, sesuai Sifat Transformasi Laplace 1, maka

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}$$

$$= \frac{1}{2}t^2$$

maka :

$$\mathcal{L}^{-1}\left\{e^{-3s} . \frac{1}{s^3}\right\} = \frac{1}{2}(t-3)^2 . U_3(t)$$

$$//$$

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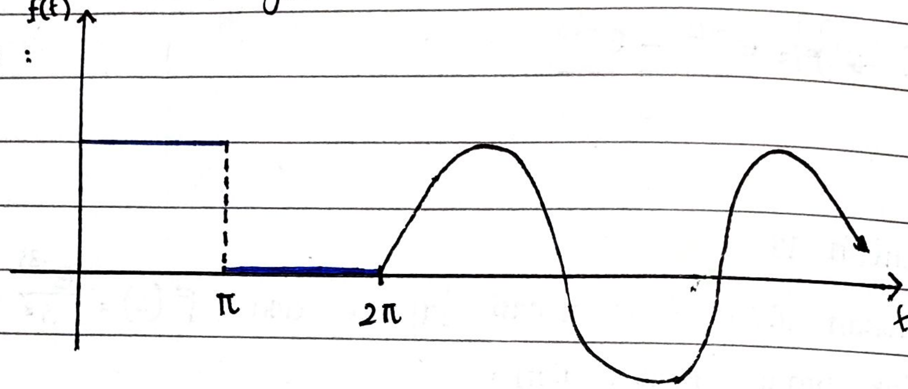
※ Contoh 19

Dapatkan transformasi laplace dari fungsi

$$f(t) = \begin{cases} 1 , & \text{jika } 0 < t < \pi \\ 0 , & \text{jika } \pi < t < 2\pi \\ \sin t , & \text{jika } t > 2\pi \end{cases}$$

Penyelesaian :



$$f(t) = [U_0(t) - U_\pi(t)] + U_{2\pi}(t) \sin(t - 2\pi)$$

(Sesuai Gambar 5.2)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{U_0(t) - U_\pi(t)\} + \mathcal{L}\{U_{2\pi}(t) . \sin(t-2\pi)\}$$

$$= \frac{e^{-0s}}{s} - \frac{e^{-\pi s}}{s} + \mathcal{L}\{U_{2\pi}(t) \sin(t-2\pi)\}$$

※ $\mathcal{L}\{U_{2\pi}(t) \sin(t-2\pi)\} = e^{-as} F(s)$

$$= e^{-2\pi s} . \mathcal{L}\{\sin(t)\}$$

$$= e^{-2\pi s} . \frac{1}{s^2+1}$$

$$= \frac{e^{-2\pi s}}{s^2+1}$$

$$\mathcal{L}\{f(t)\} = \frac{e^0 - e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$$

$$= \frac{1 - e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$$

$$= \frac{1}{s} - \frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$$

//

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Konvolusi

* Contoh 20

Bila $\mathcal{L}\{f(t)\} = \dfrac{1}{s^2(s^2+3)}$, untuk setiap $s>0$, tentukan $f(t)$.

Penyelesaian :

* Bentuk Umum : $\mathcal{L}\{(h*g)(t)\} = H(s) \cdot G(s)$

misal : anggap $G(s) = \dfrac{1}{s^2}$, sehingga $g(t) = t$. Anggap $h(s) = \dfrac{1}{s^2+3}$

Sehingga $h(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right\} = \dfrac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\dfrac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right\}$

maka, $h(t) = \dfrac{1}{\sqrt{3}}\sin\sqrt{3}t$.

$$\rightarrow \mathcal{L}\{(h*g)(t)\} = \mathcal{L}^{-1}\{H(s) * G(s)\}$$
$$= \mathcal{L}^{-1}\{H(s)\} * \mathcal{L}^{-1}\{G(s)\}$$
$$= \frac{1}{s^2+3} \cdot \frac{1}{s^2}$$
$$= \frac{1}{s^2} \cdot \frac{1}{s^2+3}$$
$$= t \cdot \frac{1}{\sqrt{3}} \sin\sqrt{3}t$$

* Sesuai Definisi 5.3 : $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$, $\forall t \geq 0$

Sehingga $\Rightarrow f(t) = \int_0^t (t-r) \cdot \dfrac{1}{\sqrt{3}} \sin\sqrt{3}r \, dr$

$$f(t) = \frac{1}{\sqrt{3}}\left[\int_0^t t\sin\sqrt{3}r\,dr - \int_0^t r\sin\sqrt{3}r\,dr\right]$$

► $\int_0^t t\sin\sqrt{3}r\,dr = t\int_0^t \sin\sqrt{3}r\,dr$, maka : $\int_0^t t\sin\sqrt{3}r\,dr = t\int_0^{\sqrt{3}t} \sin p \cdot \dfrac{dp}{\sqrt{3}}$

misal : $p = \sqrt{3}r$

$dp = \sqrt{3}\,dr$

$$= \frac{t}{\sqrt{3}}\left[-\cos p\right]_0^{\sqrt{3}t}$$

BA : $p = \sqrt{3}t$

$$= -\frac{t}{\sqrt{3}}\cos\sqrt{3}t + \frac{t}{\sqrt{3}}$$

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$$\int_0^t r\sin\sqrt{3}r\,dr = \int_0^{\sqrt{3}t} \frac{p}{\sqrt{3}} \sin p \cdot \frac{dp}{\sqrt{3}} = \frac{1}{3}\int_0^{\sqrt{3}t} p\sin p\,dp$$

misal : $p = \sqrt{3}r$

$dp = \sqrt{3}\,dr$

BA = $p = \sqrt{3}t$

misal : $m = p$ $\quad dn = \sin p\,dp$

$dm = dp$ $\quad n = -\cos p$

$$\Rightarrow \frac{1}{3}\int_0^{\sqrt{3}t} p\sin p\,dp = \frac{1}{3}\left[m.n - \int n\,dm\right]$$

$$\int_0^{\sqrt{3}t} p\sin p\,dp = -p\cos p\Big]_0^{\sqrt{3}t} - \left[\int_0^{\sqrt{3}t} -\cos p\,.\,dp\right]$$

$$= -\sqrt{3}t\cos\sqrt{3}t + 0 + \left[\sin p\right]_0^{\sqrt{3}t}$$

$$= -\sqrt{3}t\cos\sqrt{3}t + \sin\sqrt{3}t$$

maka $\int_0^t r\sin\sqrt{3}\,r\,dr = \frac{1}{3}\left[-\sqrt{3}t\cos\sqrt{3}t + \sin\sqrt{3}t\right]$

$$= -\frac{1}{3}\sqrt{3}\,t\cos\sqrt{3}t + \frac{1}{3}\sin\sqrt{3}t$$

Jadi, $\frac{1}{\sqrt{3}}\left[\int_0^t t\sin\sqrt{3}r\,dr - \int_0^t r\sin\sqrt{3}\,r\,dr\right]$

$$= \frac{1}{\sqrt{3}}\left[-\frac{t}{\sqrt{3}}\cos\sqrt{3}t + \frac{t}{\sqrt{3}} - \left(-\frac{1}{3}\sqrt{3}\,t\cos\sqrt{3}t + \frac{1}{3}\sin\sqrt{3}t\right)\right]$$

* Rasionalkan dan Komutatif kan.

$$= \frac{1}{\sqrt{3}}\left[\cancel{-\frac{t}{3}\sqrt{3}\cos\sqrt{3}t} + \cancel{\frac{t}{3}\sqrt{3}\cos\sqrt{3}t} + \frac{t}{\sqrt{3}} - \frac{1}{3}\sin\sqrt{3}t\right]$$

$$= \frac{1}{3}t - \frac{1}{3\sqrt{3}}\sin\sqrt{3}\,t$$

//

Jadi, $f(t) = \frac{1}{3}t - \frac{1}{3\sqrt{3}}\sin\sqrt{3}\,t$

//

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※ Contoh 21

Tentukan penyelesaian PD berikut : $y'' + 4y' + 13y = \frac{1}{3}e^{-2t}\sin 3t$ dengan kondisi awal $y(0) = 1$; $y'(0) = -2$

Penyelesaian :

$$\mathcal{L}\{y'' + 4y' + 13y\} = \mathcal{L}\{\tfrac{1}{3}e^{-2t}\sin 3t\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{4y'\} + \mathcal{L}\{13y\} = \mathcal{L}\{\tfrac{1}{3}e^{-2t}\sin 3t\}$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 13\mathcal{L}\{y\} = \frac{1}{3}\mathcal{L}\{e^{-2t}\sin 3t\}$$

$$[s^2F(s) - sy(0) - y'(0)] + 4[sF(s) - y(0)] + 13F(s) = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{(s+2)^2+3^2}$$

$$F(s)[s^2+4s+13] - s - (-2) + 4(-1) = \frac{1}{(s+2)^2+3^2}$$

$$F(s)[s^2+4s+13] - s - 2 = \frac{1}{(s+2)^2+3^2}$$

$$F(s)[(s+2)^2 - 4 + 13] = \frac{1}{(s+2)^2+3^2} + s + 2$$

$$F(s) = \frac{1}{[(s+2)^2+3^2]^2} + \frac{(s+2)}{(s+2)^2+3^2}$$

※ $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} = e^{-2t} . \cos 3t$

※ $\mathcal{L}^{-1}\left\{\frac{1}{[(s+2)^2+3^2]^2}\right\} = e^{-2t} . \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2} \cdot \frac{1}{s^2+3^2}\right\}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

Ingat ! $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\,\frac{3}{s^2+3^2}\right\}$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \frac{1}{3} . \sin 3t$$

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$* \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$

$= \int_0^t \frac{1}{3}\sin 3(t-r)\,\frac{1}{3}\sin 3r\,dr$

$= \frac{1}{9}\int_0^t \sin 3(t-r)\,.\sin 3r\,dr$

$*$ Ingat !

$$\begin{aligned} \cos(a-b) &= \cos a\cos b + \sin a \sin b \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \hline \cos(a-b)-\cos(a+b) &= 2\sin a \sin b \end{aligned}$$

Jadi, $\sin a \sin b = \dfrac{\cos(a-b)-\cos(a+b)}{2}$

$* \sin 3(t-r)\sin 3r = \dfrac{\cos(3t-3r-3r)-\cos(3t-3r+3r)}{2}$

$= \dfrac{\cos(3t-6r)-\cos 3t}{2}$

$\rightarrow \int_0^t \sin 3(t-r)\sin 3r\,dr = \frac{1}{2}\left[\int_0^t \cos(3t-6r)\,dr - \int_0^t \cos 3t\,dr\right]$

$* \int_0^t \cos(3t-6r)\,dr \rightarrow -\frac{1}{6}\int_0^{-3t} \cos p\,.\,dp = -\frac{1}{6}\Big[\sin p\Big]_{3t}^{-3t}$

misal : $p = 3t-6r$

$dp = -6\,dr$

BA : $p = 3t-6t$

$= -3t$

BB : $p = 3t-0$

$= 3t$

$= -\frac{1}{6}\left(\sin(-3t) - \sin 3t\right)$

$= -\frac{1}{6}\sin(-3t) + \frac{1}{6}\sin 3t$

$* \sin x = -\sin(-x)$

$= \frac{2}{6}\sin 3t$

$= \frac{1}{3}\sin 3t$

$* \int_0^t \cos(3t-6r)\,dr = \frac{1}{3}\sin 3t$ //

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$\int_0^t \cos 3t \, dr = \cos 3t \int_0^t dr$

$$= \cos 3t \, [r]_0^t$$

$$= t\cos 3t$$

$$\int_0^t \sin 3(t-r) \sin 3r \, dr = \frac{1}{2}\left[\int_0^t \cos(3t-6r)\,dr - \int_0^t \cos 3t \, dr\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin 3t - t\cos 3t\right]$$

$$= \frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t$$

$$f(t) = \frac{1}{9}\left[\int_0^t \sin 3(t-r)\sin 3r \, dr\right]$$

$$= \frac{1}{9}\left[\frac{1}{6}\sin 3t - \frac{1}{6} \cdot 3t\cos 3t\right]$$

$$= \left[\frac{1}{54}\sin 3t - \frac{3}{54}t\cos t\right]$$

$$\mathcal{L}^{-1}\left\{\frac{1}{[(s+2)^2+3^2]^2}\right\} = e^{-2t}\left[\frac{1}{54}\sin 3t - \frac{3}{54}t\cos t\right]$$

$$f(t) = e^{-2t}\cos 3t + e^{-2t}\left[\frac{1}{54}\sin 3t - \frac{3}{54}t\cos t\right]$$

$$= e^{-2t}\left[\frac{1}{54}\sin 3t - \frac{3t\cos t}{54} + \cos 3t\right]$$